The Ivy League Education Center

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MOCK EXAMINATION

AMC 8

American Mathematics Contest 8

Test Sample

Detailed Solutions

AMC 8 Mock Test

Detailed Solutions

Problem 1

Answer: **(E)**

Solution 1

Pairing the first two terms, the next two terms, etc. yields

$$1 - 2 + 3 - 4 + \cdots - 2020 + 2021 = (1-2) + (3-4) + \cdots + (2019 - 2020) + 2021$$
$$= \underbrace{-1 - 1 - 1 - \cdots - 1}_{1010} + 2021 = 2021 - 1010 = 1011,$$

since there are 1011 of the -1's.

Solution 2

$$1 + \big((-2+3) + (-4+5) + \cdots + (-2020 + 2021)\big) = 1 + (\underbrace{1 + 1 + \cdots + 1}_{1010})$$

$$= 1 + 1010 = 1011.$$

Problem 2

Answer: **(D)**

Solution:

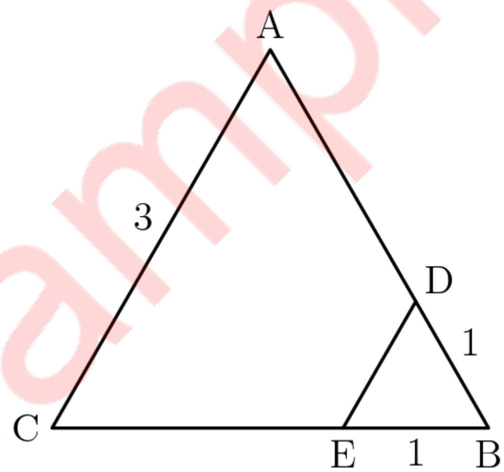
The desired number is the arithmetic average or mean:

$$\frac{\frac{1}{16} + \frac{1}{20}}{2} = \frac{\frac{1}{4} + \frac{1}{5}}{2 \cdot 4} = \frac{\frac{9}{20}}{2 \cdot 4} = \frac{9}{160}.$$

Problem 3

Answer: **(E)**

Solution:



Because $DB = EB$ and $\angle DBE = 60°$, it follows that ΔDBE is also equilateral. Thus,

$$DE = 1.$$

The perimeter of ΔABC is 9. We delete two segments of length 1, DB and EB, and add a segment of length 1, DE.

Hence, the perimeter of quadrilateral $ADEC$ is

$$9 - 2 + 1 = 8.$$

Problem 4

Answer: (D)

Solution:

The number of chunks in 240 blocks is:

$$240 \cdot 512.$$

Divide this by 120 to determine the number of seconds necessary to transmit. So we have:

$$\frac{240 \cdot 512}{120} = 1024 \text{ seconds} \approx 27 \text{ minutes}.$$

Problem 5

Answer: **(B)**

Solution:

Note that

$$b = 3a$$

and

$$c = 4b = 4(3a) = 12a.$$

Hence,

$$\frac{a+b}{b+c} = \frac{a+3a}{3a+12a} = \frac{4}{15}.$$

Problem 6

Answer: **(A)**

Solution:

Since Alan ate 10% of the jellybeans remaining each day, 90% of the jellybeans are left at the end of each day.

If x is the number of jellybeans in the jar originally, then

$$0.9^2 \cdot x = 81.$$

Hence,

$$x = 100.$$

Problem 7

Answer: **(D)**

Solution:

There are 2 unit cubes along each of the 4 edges of the top face of the original cube, for

$$2 \times 4 = 8$$

unit cubes that each have exactly two blue faces.

There are 3 unit cubes along each of the 4 vertical edges of the original cube, for

$$3 \times 4 = 12$$

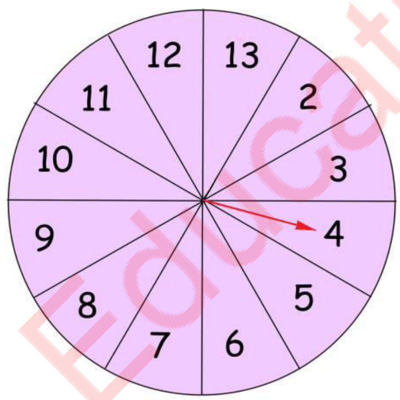
unit cubes that each have exactly two blue faces.

Hence, there is a total of

$$8 + 12 = 20$$

unit cubes that each have exactly two blue faces.

Problem 8



Answer: **(D)**

Solution:

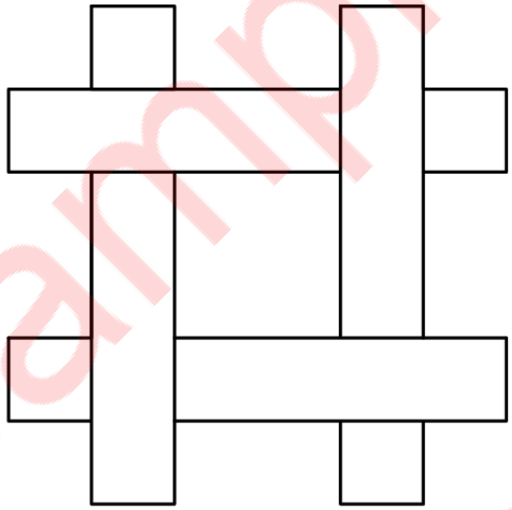
On the spinner given, the odd prime numbers are

$$3, 5, 7, 11, \text{and } 13.$$

Since the spinner is divided into 12 equal sections, the probability that the arrow stops in a section containing an add prime number is:

$$\frac{5}{12}.$$

Problem 9



Answer: **(A)**

Solution:

There are 4 strips, each has area

$$6 \cdot 1 = 6.$$

Also, 4 unit squares of side 1 are covered twice.

Hence, the total area covered is:

$$4 \cdot 6 - 4 \cdot 1 = 20.$$

Problem 10

Answer: **(B)**

Solution:

If the suggested retail price was x, then the marked price was

$$0.6x.$$

Half of this is

$$\frac{0.6x}{2} = 0.3x,$$

so Joe paid 30% of the suggested retail price.

Problem 11

Answer: **(C)**

Solution:

Note that

$$\begin{aligned} 20^{2019}\cdot 50^{2021} &= (2^{2019}\cdot 10^{2019})\cdot(5^{2019}\cdot 5^2\cdot 10^{2021}) \\ &= 5^2\cdot(2^{2019}\cdot 5^{2019})\cdot 10^{2019}\cdot 10^{2021} \\ &= 25\cdot 10^{5059} = 25\underbrace{00\cdots 00}_{5059\text{ zeros}}. \end{aligned}$$

Hence, the sum of the digits is 7.

Problem 12

Answer: **(E)**

Solution:

The subtraction problem posed is equivalent to the addition problem

$$\begin{array}{cccc} & 4 & 8 & b \\ + & c & 7 & 3 \\ \hline & 7 & a & 2 \end{array}$$

which is easier to solve. Since

$$b+3=12,$$

b must be 9. Since

$$1+8+7$$

has units digit a, a must be 6.

Because

$$1+4+c=7,$$

it follows that

$$c = 2.$$

Hence,

$$a + b + c = 6 + 9 + 2 = 17.$$

Problem 13

Answer: **(D)**

Solution:

Dividing both sides of the original equation by 3^{2020} gives:

$$1 - 3 - 3^2 + 3^3 = m,$$

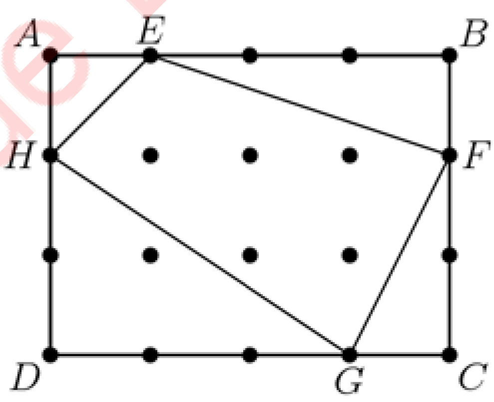
which implies that

$$m = 1 - 3 - 9 + 27 = 16.$$

Problem 14

Answer: **(E)**

Solution 1



The quadrilateral is inscribed in a 4×3 grid rectangle. Four triangular regions are inside the rectangle but outside the quadrilateral. The area of the upper-left triangle is:

$$\frac{1 \times 2}{2} = 1.$$

The area of the lower-left triangle is:

$$\frac{3 \times 1}{2} = \frac{3}{2}.$$

The area of the lower-right triangle is:

$$\frac{1 \times 1}{2} = \frac{1}{2}.$$

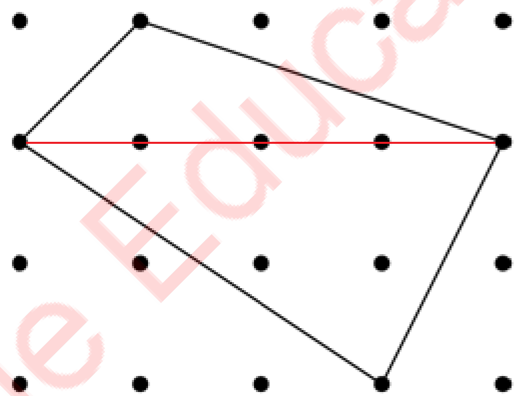
The area of the upper-right triangle is:

$$\frac{3 \times 2}{2} = 3.$$

Thus, the area of the quadrilateral is

$$12 - 1 - \frac{3}{2} - \frac{1}{2} - 3 = 6.$$

Solution 2



The quadrilateral can be partitioned into two triangles as indicated in the diagram above: the upper triangle with area

$$\frac{4 \times 2}{2} = 4,$$

and the lower triangle with area

$$\frac{4 \times 1}{2} = 2.$$

Thus, the area of the quadrilateral is

$$4 + 2 = 6.$$

Problem 15

Answer: (C)

Solution

Let x be the original number. Then moving the decimal point 4 places to the right is the same as multiplying x by 10,000. That is,

$$10{,}000x = 9 \cdot \frac{1}{x}$$

which is equivalent to

$$x^2 = \frac{4}{10{,}000}.$$

Since x is positive, it follows that

$$x = \frac{2}{100} = 0.03.$$

Problem 16

Answer: (C)

Solution

Factor each term in the product as a difference of two squares, and group the terms according to signs to get:

$$\left(\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{2020}\right)\left(1-\frac{1}{2021}\right)\right)\left(\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{2020}\right)\left(1+\frac{1}{2021}\right)\right)$$

$$= \left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{2019}{2020}\cdot\frac{2020}{2021}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{2021}{2020}\cdot\frac{2022}{2021}\right)$$

$$= \left(\frac{1}{2021}\right)\left(\frac{2022}{2}\right) = \frac{1011}{2021}.$$

Problem 17

Answer: **(A)**

Solution

We have to determine the time 100 hours before 5 p.m. Friday. Note that there are 24 hours in 1 day and

$$100 = 4 \times 24 + 4.$$

Thus, 100 hours is equal to 4 days plus 4 hours. Starting at 5 p.m. Friday, we move 4 hours back in time to 1 p.m. Friday and then an additional 4 days back in time to 1 p.m. Monday.

Hence, Jim turned his computer on at 1 p.m. Monday.

Problem 18

Answer: **(A)**

Solution

Note that the prime factorization of 100 is:

$$100 = 2 \times 2 \times 5 \times 5.$$

We are looking for three positive integers $a, b,$ and c such that

$$abc = 2 \times 2 \times 5 \times 5,$$

and whose sum $a + b + c$ is as small as possible.

A case-by-case analysis shows that $a + b + c$ is smallest when two of the numbers $a, b,$ and c are 5 and the third is 4.

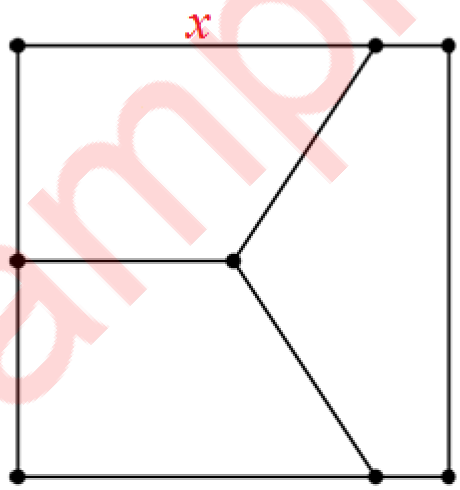
Hence, the answer is

$$a + b + c = 4 + 5 + 5 = 14.$$

Note:

To minimize $a + b + c$, we want $a, b,$ and c to be as close as possible.

Problem 19



Answer: **(D)**

Solution

Since the area of the square is 1, it follows that the area of each trapezoid is $1/3$.

Note that each trapezoid has shorter base $1/2$, long base x, and height $1/2$. So

$$\frac{\frac{1}{2}+x}{2}\cdot\frac{1}{2}=\frac{1}{3}.$$

Simplifying yields

$$\frac{1}{2}+x=\frac{4}{3},$$

and it follows that

$$x=\frac{5}{6}\,.$$

Problem 20

Answer: **(A)**

Solution

Let m be the mean of the three numbers. Then the least of the numbers is $m - 20$ and the greatest is $m + 10$. The middle of the three numbers is the median, 15. So

$$\big((m-20) + 2m + 15 - 20 + 10 + (m+10)\big) = 3m,$$

$$\frac{1}{3}\big((m-20) + 15 + (m+10)\big) = m,$$

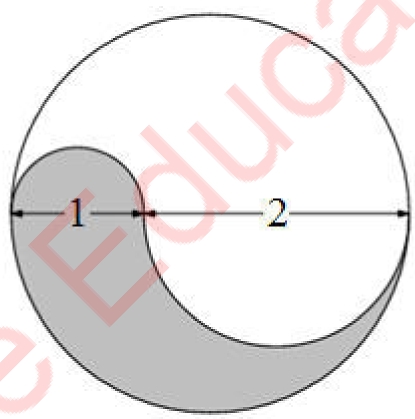
which implies that

$$m = 5.$$

Hence, the sum of the three numbers is

$$3 \cdot 5 = 15.$$

Problem 21



Answer: **(D)**

Solution

The unshaded region of a semicircle of diameter $1 + 2$ with the addition of a semicircle of diameter 2 and the deletion of a semicircle of diameter 1. Thus, the unshaded area is:

$$\frac{1}{2}\left(\pi\left(\frac{1+2}{2}\right)^2 + \pi\left(\frac{2}{2}\right)^2 - \pi\left(\frac{1}{2}\right)^2\right) = \frac{3}{2}\pi.$$

Note that the area of the circle of diameter 3 is

$$\pi\left(\frac{1+2}{2}\right)^2=\frac{9}{4}\pi.$$

So the shaded area is:

$$\frac{9}{4}\pi-\frac{3}{2}\pi=\frac{3}{4}\pi.$$

Hence, the ratio of the area, of the unshaded region to that of the shaded region is:

$$\frac{\frac{3}{2}\pi}{\frac{3}{4}\pi}=2.$$

Problem 22

Answer: **(B)**

Solution

We use casework to solve this problem.

Case 1: One child gets 1 toy, and the other two get each two toys.

There are

$$\binom{5}{2}$$

ways to select 2 toys from the 5 different toys, leaving 3 toys free. There are

$$\binom{3}{2}$$

ways to select 2 toys from the remaining 3 different toys, leaving 1 toy to be uniquely determined.

Thus, there are

$$\binom{5}{2}\binom{3}{2}=\frac{5\cdot 4}{2}\cdot\frac{3\cdot 2}{2}=30$$

ways to partition the 5 different toys into 3 groups such that two groups have each 2 toys, and the additional group has 1 toy.

There are 3 ways to assign the 3 groups to the 3 children such that each child gets exactly one group of toys, as shown below.

Child A	Child B	Child C
1	2	2
2	1	2
2	2	1

So there are

$$30 \cdot 3 = 90$$

ways to distribute the 5 different toys among 3 children in this case.

Case 2: One child gets 3 toys, and the other two get each 1 toy.

There are

$$\binom{5}{3}$$

ways to select 3 toys from the 5 different toys, leaving 2 toys free. There are

$$\binom{2}{1}$$

ways to select 1 toy from the remaining 2 different toys, leaving 1 toy to be uniquely determined.

Thus, there are

$$\binom{5}{3}\binom{2}{1} = \frac{5 \cdot 4}{2} \cdot 2 = 20$$

ways to partition the 5 different toys into 3 groups such that one group has 3 toys, and the other two groups have each 1 toy.

There are 3 ways to assign the 3 groups to the 3 children such that each child gets exactly one group of toys, as shown below.

Child A	Child B	Child C
1	2	2
2	1	2
2	2	1

So there are

$$20 \cdot 3 = 60$$

ways to distribute the 5 different toys among 3 children in this case.

Hence, there is a total of

$$90 + 60 = 150$$

ways to distribute the 5 different toys among 3 children such that each one gets at least one toy.

Problem 23

Answer: **(C)**

Solution

Let N be a positive integer and d be a divisor of N. Then

$$\frac{N}{d}$$

is also a factor of N. Thus, the divisors of N occur in pairs

$$d, \qquad \frac{N}{d}$$

and these two divisors are distinct unless N is a perfect square and

$$d = \sqrt{N}.$$

It follows that N has an odd number of positive integer divisors if and only if N is a perfect square.

Note that

$$45^2 = 2025$$

and

$$44^2 = 1936.$$

Hence, there are 44 perfect squares among the numbers $1, 2, 3, \cdots, 2021$.

Problem 24

Answer: **(D)**

Solution

Let the total number of students be $3n$. Then there are $2n$ boys and n girls.

Since there are n boys to join the math club, and $\frac{2n}{3}$ girls to join the math club, it follows that there is a total of

$$n + \frac{2n}{3} = \frac{5n}{3}$$

students to join the math club.

Hence, the probability that the student selected is a boy is:

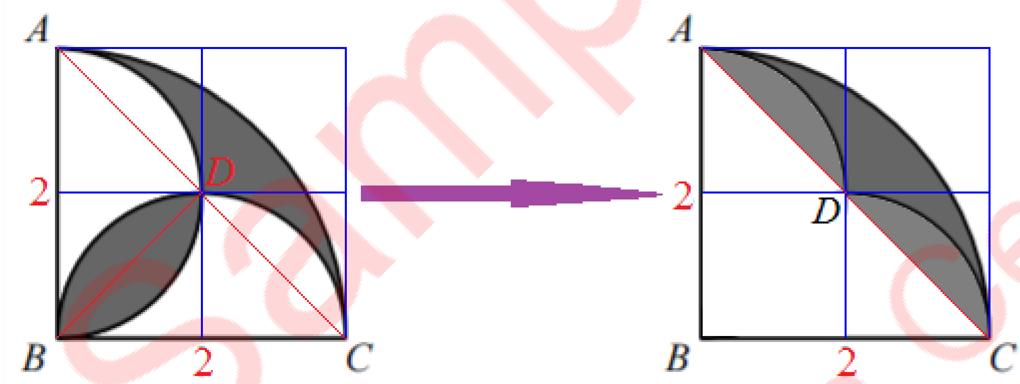
$$\frac{n}{\frac{5n}{3}} = \frac{3}{5}.$$

Problem 25

Answer: **(B)**

Solution 1:

Let D be the point of intersection of the two congruent semicircles. Connect BD to partition the intersection of the interiors of two semicircles into two congruent circular segments.

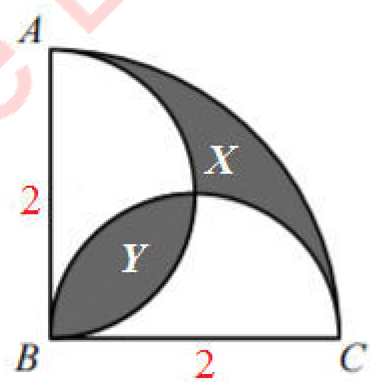


Rotate the bottom half circular segment 90 degrees counterclockwise about D, and rotate the top half circular segment 90 degrees clockwise about D, to create a larger circular segment.

Hence, the area of the shaded region equals subtracting the area of the isosceles right triangle with legs of length 2 from the area of the quarter-circle of radius 2, which is

$$\frac{1}{4}\pi \cdot 2^2 - \frac{2 \times 2}{2} = \pi - 2.$$

Solution 2:



Let X be the area of the shaded region that lies outside of both semicircles, and let Y be the area of the shaded region that lies inside of both semicircles.
The sum of the areas of both semicircles counts the shaded area Y twice since the area of overlap of the semicircles is Y.

Subtracting Y from the sum of the areas of both semicircles, and add X, gets the area of the quarter-circle ABC. That is, the area of quarter-circle ABC equals the sum of the area of the semicircle with diameter AB and the area of the semicircle with diameter BC.

The area of quarter-circle ABC is

$$\frac{1}{4}\pi \cdot 8^2 = 16\pi.$$

The area of the semicircle drawn on AB or BC is

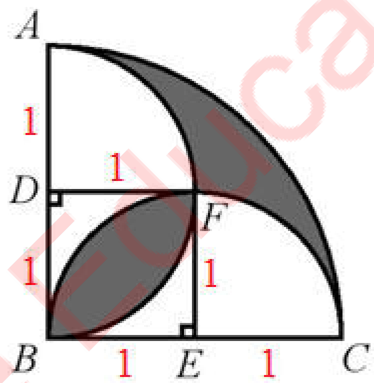
$$\frac{1}{2}\pi \cdot 4^2 = 8\pi.$$

Thus,

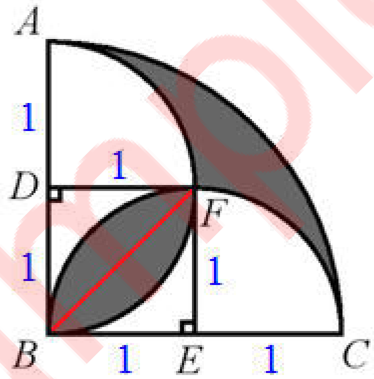
$$16\pi = 8\pi + 8\pi - Y + X,$$

and so

$$X = Y.$$



Now let D be the midpoint of AB, and E be the midpoint of BC. Draw square $DBEF$. Then DF is a tangent of the semicircle drawn on BC and EF is a tangent of the semicircle drawn on AB. So F must be the point of intersection of the two semicircles.



Construct BF, which is the diagonal of square $DBEF$. By symmetry, BF divides the shaded area Y into two equal areas. Each of these equal areas, $\frac{Y}{2}$, is equal to the area of isosceles right ΔBEF with legs of length 1 subtracted from the area of the quarter-circle BEF with radius 1. That is,

$$\frac{Y}{2} = \frac{1}{4}\pi \cdot 1^2 - \frac{1 \times 1}{2} = \frac{\pi}{4} - \frac{1}{2},$$

and so

$$Y = \frac{\pi}{2} - 1.$$

Hence, the area of the shaded region is

$$X + Y = 2Y = \pi - 2.$$